

Motivic Milnor fibers and Jordan normal forms of Milnor monodromies *

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Abstract

By calculating the equivariant mixed Hodge numbers of motivic Milnor fibers introduced by Denef-Loeser, we obtain explicit formulas for the Jordan normal forms of Milnor monodromies. The numbers of the Jordan blocks will be described by the Newton polyhedron of the polynomial.

1 Introduction

In this paper, by using motivic Milnor fibers introduced by Denef-Loeser [4] and [5], we obtain explicit formulas for the Jordan normal forms of Milnor monodromies. Let $f(x) = \sum_{v \in \mathbb{Z}_+^n} a_v x^v \in \mathbb{C}[x_1, \dots, x_n]$ be a polynomial on \mathbb{C}^n such that the hypersurface $f^{-1}(0) = \{x \in \mathbb{C}^n \mid f(x) = 0\}$ has an isolated singular point at $0 \in \mathbb{C}^n$. Then by a fundamental theorem of Milnor [15], the Milnor fiber F_0 of f at $0 \in \mathbb{C}^n$ has the homotopy type of bouquet of $(n-1)$ -spheres. In particular, we have $H^j(F_0; \mathbb{C}) \simeq 0$ ($j \neq 0, n-1$). Denote by

$$\Phi_{n-1,0}: H^{n-1}(F_0; \mathbb{C}) \xrightarrow{\sim} H^{n-1}(F_0; \mathbb{C}) \quad (1.1)$$

the $(n-1)$ -th Milnor monodromy of f at $0 \in \mathbb{C}^n$. By the theory of monodromy zeta functions due to A'Campo [1] and Varchenko [26] etc., the eigenvalues of $\Phi_{n-1,0}$ were fairly well-understood. See Oka's book [17] for an excellent exposition of this very important result. However to the best of our knowledge, it seems that the Jordan normal form of $\Phi_{n-1,0}$ is not fully understood yet. In this paper, we give a combinatorial description of the Jordan normal form of $\Phi_{n-1,0}$ by using motivic Milnor fibers (For a computer algorithm by Brieskorn lattices, see Schulze [22] etc.).

From now on, let us assume also that f is convenient and non-degenerate at $0 \in \mathbb{C}^n$ (see Definitions 4.1 and 4.2). Note that the second condition is satisfied by generic polynomials f . Then we can describe the Jordan normal form of $\Phi_{n-1,0}$ very explicitly as follows. We call the convex hull of $\bigcup_{v \in \text{supp}(f)} \{v + \mathbb{R}_+^n\}$ in \mathbb{R}_+^n the Newton polyhedron of f and denote it by $\Gamma_+(f)$. Let q_1, \dots, q_l (resp. $\gamma_1, \dots, \gamma_{l'}$) be the 0-dimensional (resp. 1-dimensional) faces of $\Gamma_+(f)$ such that $q_i \in \text{Int}(\mathbb{R}_+^n)$ (resp. the relative interior $\text{rel.int}(\gamma_i)$ of γ_i is contained in $\text{Int}(\mathbb{R}_+^n)$). For each q_i (resp. γ_i), denote by $d_i > 0$ (resp. $e_i > 0$) its lattice distance

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$\text{dist}(q_i, 0)$ (resp. $\text{dist}(\gamma_i, 0)$) from the origin $0 \in \mathbb{R}^n$. For $1 \leq i \leq l'$, let Δ_i be the convex hull of $\{0\} \sqcup \gamma_i$ in \mathbb{R}^n . Then for $\lambda \in \mathbb{C} \setminus \{1\}$ and $1 \leq i \leq l'$ such that $\lambda^{e_i} = 1$ we set

$$n(\lambda)_i = \#\{v \in \mathbb{Z}^n \cap \text{rel.int}(\Delta_i) \mid \text{ht}(v, \gamma_i) = k\} + \#\{v \in \mathbb{Z}^n \cap \text{rel.int}(\Delta_i) \mid \text{ht}(v, \gamma_i) = e_i - k\}, \quad (1.2)$$

where k is the minimal positive integer satisfying $\lambda = \zeta_{e_i}^k$ ($\zeta_{e_i} := \exp(2\pi\sqrt{-1}/e_i)$) and for $v \in \mathbb{Z}^n \cap \text{rel.int}(\Delta_i)$ we denote by $\text{ht}(v, \gamma_i)$ the lattice height of v from the base γ_i of Δ_i . Then in Section 4 we prove the following result which describes the number of Jordan blocks for each fixed eigenvalue $\lambda \neq 1$ in $\Phi_{n-1,0}$. Recall that by the monodromy theorem the sizes of such Jordan blocks are bounded by n .

Theorem 1.1 *Assume that f is convenient and non-degenerate at $0 \in \mathbb{C}^n$. Then for any $\lambda \in \mathbb{C}^* \setminus \{1\}$ we have*

- (i) *The number of the Jordan blocks for the eigenvalue λ with the maximal possible size n in $\Phi_{n-1,0}: H^{n-1}(F_0; \mathbb{C}) \xrightarrow{\sim} H^{n-1}(F_0; \mathbb{C})$ is equal to $\#\{q_i \mid \lambda^{d_i} = 1\}$.*
- (ii) *The number of the Jordan blocks for the eigenvalue λ with size $n-1$ in $\Phi_{n-1,0}$ is equal to $\sum_{i: \lambda^{e_i}=1} n(\lambda)_i$.*

Namely the Jordan blocks for the eigenvalues $\lambda \neq 1$ in the monodromy $\Phi_{n-1,0}$ are determined by the lattice distances of the faces of $\Gamma_+(f)$ from the origin $0 \in \mathbb{R}^n$. The monodromy theorem asserts also that the sizes of the Jordan blocks for the eigenvalue 1 in $\Phi_{n-1,0}$ are bounded by $n-1$. In this case, we have the following result. Denote by Π_f the number of the lattice points on the 1-skeleton of $\partial\Gamma_+(f) \cap \text{Int}(\mathbb{R}_+^n)$. For a compact face $\gamma \prec \Gamma_+(f)$, denote by $l^*(\gamma)$ the number of the lattice points on the relative interior $\text{rel.int}(\gamma)$ of γ .

Theorem 1.2 *In the situation of Theorem 1.1 we have*

- (i) (van Doorn-Steenbrink [6]) *The number of the Jordan blocks for the eigenvalue 1 with the maximal possible size $n-1$ in $\Phi_{n-1,0}$ is Π_f .*
- (ii) *The number of the Jordan blocks for the eigenvalue 1 with size $n-2$ in $\Phi_{n-1,0}$ is equal to $2 \sum_{\gamma} l^*(\gamma)$, where γ ranges through the compact faces of $\Gamma_+(f)$ such that $\dim \gamma = 2$ and $\text{rel.int}(\gamma) \subset \text{Int}(\mathbb{R}_+^n)$. In particular, this number is even.*

Note that Theorem 1.2 (i) was previously obtained in van Doorn-Steenbrink [6] by using different methods. Roughly speaking, the nilpotent part for the eigenvalue 1 in the monodromy $\Phi_{n-1,0}$ is determined by the convexity of the hypersurface $\partial\Gamma_+(f) \cap \text{Int}(\mathbb{R}_+^n)$. Thus Theorems 1.1 and 1.2 generalize the well-known fact that the monodromies of quasi-homogeneous polynomials are semisimple. In fact, by our results in Sections 2 and 4 a general algorithm for computing all the spectral pairs of the Milnor fiber F_0 is obtained. This in particular implies that we can compute the Jordan normal form of $\Phi_{n-1,0}$ completely. Note that the spectrum of F_0 obtained in Saito [20] and Varchenko-Khovanskii [27] is not enough to deduce the Jordan normal form. Moreover, if any compact face of $\Gamma_+(f)$ is prime (see Definition 2.9) we obtain also a closed formula for the Jordan normal form. See Section 4 for the details.

This paper is organized as follows. In Section 2, we introduce some generalizations of the results of Danilov-Khovanskii [3] obtained in [14]. By them we obtain a general algorithm for computing the equivariant mixed Hodge numbers of non-degenerate toric hypersurfaces. In Section 3, we recall some basic definitions and results on motivic Milnor fibers introduced by Denef-Loeser [4] and [5]. Then in Section 4, by rewriting them in terms of the Newton polyhedron $\Gamma_+(f)$ with the help of the results in Section 2 and [14], we prove various combinatorial formulas for the Jordan normal form of the Milnor monodromy $\Phi_{n-1,0}$. Although our proof for the eigenvalue 1 in this paper is very different from the one in [14], our results in Section 4 are completely parallel to those for monodromies at infinity obtained in [14]. We thus find a striking symmetry between local and global. Finally, let us mention that in [7] the results for the other eigenvalues $\lambda \neq 1$ in this paper were already generalized to the monodromies over complete intersection subvarieties in \mathbb{C}^n .

2 Preliminary notions and results

In this section, we recall our results in [14, Section 2] which will be used in this paper. They are slight generalizations of the results in Danilov-Khovanskii [3].

Definition 2.1 Let $g(x) = \sum_{v \in \mathbb{Z}^n} a_v x^v$ ($a_v \in \mathbb{C}$) be a Laurent polynomial on $(\mathbb{C}^*)^n$.

- (i) We call the convex hull of $\text{supp}(g) := \{v \in \mathbb{Z}^n \mid a_v \neq 0\} \subset \mathbb{Z}^n$ in \mathbb{R}^n the Newton polytope of g and denote it by $NP(g)$.
- (ii) For $u \in (\mathbb{R}^n)^*$, we set $\Gamma(g; u) := \{v \in NP(g) \mid \langle u, v \rangle = \min_{w \in NP(g)} \langle u, w \rangle\}$.
- (iii) For $u \in (\mathbb{R}^n)^*$, we define the u -part of g by $g^u(x) := \sum_{v \in \Gamma(g; u)} a_v x^v$.

Definition 2.2 ([9]) Let g be a Laurent polynomial on $(\mathbb{C}^*)^n$. Then we say that the hypersurface $Z^* = \{x \in (\mathbb{C}^*)^n \mid g(x) = 0\}$ of $(\mathbb{C}^*)^n$ is non-degenerate if for any $u \in (\mathbb{R}^n)^*$ the hypersurface $\{x \in (\mathbb{C}^*)^n \mid g^u(x) = 0\}$ is smooth and reduced.

In the sequel, let us fix an element $\tau = (\tau_1, \dots, \tau_n) \in T := (\mathbb{C}^*)^n$ and let g be a Laurent polynomial on $(\mathbb{C}^*)^n$ such that $Z^* = \{x \in (\mathbb{C}^*)^n \mid g(x) = 0\}$ is non-degenerate and invariant by the automorphism $l_\tau: (\mathbb{C}^*)^n \xrightarrow{\tau \times} (\mathbb{C}^*)^n$ induced by the multiplication by τ . Set $\Delta = NP(g)$ and for simplicity assume that $\dim \Delta = n$. Then there exists $\beta \in \mathbb{C}$ such that $l_\tau^* g = g \circ l_\tau = \beta g$. This implies that for any vertex v of $\Delta = NP(g)$ we have $\tau^v = \tau_1^{v_1} \cdots \tau_n^{v_n} = \beta$. Moreover by the condition $\dim \Delta = n$ we see that $\tau_1, \tau_2, \dots, \tau_n$ are roots of unity. For $p, q \geq 0$ and $k \geq 0$, let $h^{p,q}(H_c^k(Z^*; \mathbb{C}))$ be the mixed Hodge number of $H_c^k(Z^*; \mathbb{C})$ and set

$$e^{p,q}(Z^*) = \sum_k (-1)^k h^{p,q}(H_c^k(Z^*; \mathbb{C})) \quad (2.1)$$

as in [3]. The above automorphism of $(\mathbb{C}^*)^n$ induces a morphism of mixed Hodge structures $l_\tau^*: H_c^k(Z^*; \mathbb{C}) \xrightarrow{\sim} H_c^k(Z^*; \mathbb{C})$ and hence \mathbb{C} -linear automorphisms of the (p, q) -parts $H_c^k(Z^*; \mathbb{C})^{p,q}$ of $H_c^k(Z^*; \mathbb{C})$. For $\alpha \in \mathbb{C}$, let $h^{p,q}(H_c^k(Z^*; \mathbb{C}))_\alpha$ be the dimension of the α -eigenspace $H_c^k(Z^*; \mathbb{C})_\alpha^{p,q}$ of this automorphism of $H_c^k(Z^*; \mathbb{C})^{p,q}$ and set

$$e^{p,q}(Z^*)_\alpha = \sum_k (-1)^k h^{p,q}(H_c^k(Z^*; \mathbb{C}))_\alpha. \quad (2.2)$$

We call $e^{p,q}(Z^*)_\alpha$ the equivariant mixed Hodge numbers of Z^* . Since we have $l'_\tau = \text{id}_{Z^*}$ for some $r \gg 0$, these numbers are zero unless α is a root of unity. Obviously we have

$$e^{p,q}(Z^*) = \sum_{\alpha \in \mathbb{C}} e^{p,q}(Z^*)_\alpha, \quad e^{p,q}(Z^*)_\alpha = e^{q,p}(Z^*)_{\bar{\alpha}}. \quad (2.3)$$

In this setting, along the lines of Danilov-Khovanskii [3] we can give an algorithm for computing these numbers $e^{p,q}(Z^*)_\alpha$ as follows. First of all, as in [3, Section 3] we have the following result.

Proposition 2.3 ([14, Proposition 2.6]) *For $p, q \geq 0$ such that $p+q > n-1$, we have*

$$e^{p,q}(Z^*)_\alpha = \begin{cases} (-1)^{n+p+1} \binom{n}{p+1} & (\alpha = 1 \text{ and } p = q), \\ 0 & (\text{otherwise}), \end{cases} \quad (2.4)$$

(we used the convention $\binom{a}{b} = 0$ ($0 \leq a < b$) for binomial coefficients).

For a vertex w of Δ , consider the translated polytope $\Delta^w := \Delta - w$ such that $0 \prec \Delta^w$ and $\tau^v = 1$ for any vertex v of Δ^w . Then for $\alpha \in \mathbb{C}$ and $k \geq 0$ set

$$l^*(k\Delta)_\alpha = \#\{v \in \text{Int}(k\Delta^w) \cap \mathbb{Z}^n \mid \tau^v = \alpha\} \in \mathbb{Z}_+ := \mathbb{Z}_{\geq 0}. \quad (2.5)$$

We can easily see that these numbers $l^*(k\Delta)_\alpha$ do not depend on the choice of the vertex w of Δ . We define a formal power series $P_\alpha(\Delta; t) = \sum_{i \geq 0} \varphi_{\alpha,i}(\Delta) t^i$ by

$$P_\alpha(\Delta; t) = (1-t)^{n+1} \left\{ \sum_{k \geq 0} l^*(k\Delta)_\alpha t^k \right\}. \quad (2.6)$$

Then we can easily show that $P_\alpha(\Delta; t)$ is actually a polynomial as in [3, Section 4.4].

Theorem 2.4 ([14, Theorem 2.7]) *In the situation as above, we have*

$$\sum_q e^{p,q}(Z^*)_\alpha = \begin{cases} (-1)^{p+n+1} \binom{n}{p+1} + (-1)^{n+1} \varphi_{\alpha, n-p}(\Delta) & (\alpha = 1), \\ (-1)^{n+1} \varphi_{\alpha, n-p}(\Delta) & (\alpha \neq 1). \end{cases} \quad (2.7)$$

By Proposition 2.3 and Theorem 2.4 we can now calculate the numbers $e^{p,q}(Z^*)_\alpha$ on the non-degenerate hypersurface $Z^* \subset (\mathbb{C}^*)^n$ for any $\alpha \in \mathbb{C}$ as in [3, Section 5.2]. Indeed for a projective toric compactification X of $(\mathbb{C}^*)^n$ such that the closure $\overline{Z^*}$ of Z^* in X is smooth, the variety $\overline{Z^*}$ is smooth projective and hence there exists a perfect pairing

$$H^{p,q}(\overline{Z^*}; \mathbb{C})_\alpha \times H^{n-1-p, n-1-q}(\overline{Z^*}; \mathbb{C})_{\alpha^{-1}} \longrightarrow \mathbb{C} \quad (2.8)$$

for any $p, q \geq 0$ and $\alpha \in \mathbb{C}^*$ (see for example [28, Section 5.3.2]). Therefore, we obtain equalities $e^{p,q}(\overline{Z^*})_\alpha = e^{n-1-p, n-1-q}(\overline{Z^*})_{\alpha^{-1}}$ which are necessary to proceed the algorithm in [3, Section 5.2]. We have also the following analogue of [3, Proposition 5.8].

Proposition 2.5 ([14, Proposition 2.8]) *For any $\alpha \in \mathbb{C}$ and $p > 0$ we have*

$$e^{p,0}(Z^*)_\alpha = e^{0,p}(Z^*)_{\bar{\alpha}} = (-1)^{n-1} \sum_{\substack{\Gamma \prec \Delta \\ \dim \Gamma = p+1}} l^*(\Gamma)_\alpha. \quad (2.9)$$

The following result is an analogue of [3, Corollary 5.10]. For $\alpha \in \mathbb{C}$, denote by $\Pi(\Delta)_\alpha$ the number of the lattice points $v = (v_1, \dots, v_n)$ on the 1-skeleton of $\Delta^w = \Delta - w$ such that $\tau^v = \alpha$, where w is a vertex of Δ .

Proposition 2.6 ([14, Proposition 2.9]) *In the situation as above, for any $\alpha \in \mathbb{C}^*$ we have*

$$e^{0,0}(Z^*)_\alpha = \begin{cases} (-1)^{n-1} (\Pi(\Delta)_1 - 1) & (\alpha = 1), \\ (-1)^{n-1} \Pi(\Delta)_{\alpha^{-1}} & (\alpha \neq 1). \end{cases} \quad (2.10)$$

For a vertex w of Δ , we define a closed convex cone $\text{Con}(\Delta, w)$ by $\text{Con}(\Delta, w) = \{r \cdot (v - w) \mid r \in \mathbb{R}_+, v \in \Delta\} \subset \mathbb{R}^n$.

Definition 2.7 ([3]) Let Δ and Δ' be two n -dimensional integral polytopes in $(\mathbb{R}^n, \mathbb{Z}^n)$. We denote by $\text{som}(\Delta)$ (resp. $\text{som}(\Delta')$) the set of vertices of Δ (resp. Δ'). Then we say that Δ' majorizes Δ if there exists a map $\Psi: \text{som}(\Delta') \rightarrow \text{som}(\Delta)$ such that $\text{Con}(\Delta, \Psi(w)) \subset \text{Con}(\Delta', w)$ for any vertex w of Δ' .

For an integral polytope Δ in $(\mathbb{R}^n, \mathbb{Z}^n)$, we denote by X_Δ the toric variety associated with the dual fan of Δ (see Fulton [8] and Oda [16] etc.). Recall that if Δ' majorizes Δ there exists a natural morphism $X_{\Delta'} \rightarrow X_\Delta$.

Proposition 2.8 ([14, Proposition 2.12]) *Let Δ and $Z_\Delta^* = Z^*$ with an action of l_τ be as above. Assume that an n -dimensional integral polytope Δ' in $(\mathbb{R}^n, \mathbb{Z}^n)$ majorizes Δ by the map $\Psi: \text{som}(\Delta') \rightarrow \text{som}(\Delta)$. Then for the closure $\overline{Z^*}$ of Z^* in $X_{\Delta'}$ we have*

$$\begin{aligned} \sum_q e^{p,q}(\overline{Z^*})_1 &= \sum_{\Gamma \prec \Delta'} (-1)^{\dim \Gamma + p + 1} \left\{ \binom{\dim \Gamma}{p+1} - \binom{b_\Gamma}{p+1} \right\} \\ &\quad + \sum_{\Gamma \prec \Delta'} (-1)^{\dim \Gamma + 1} \sum_{i=0}^{\min\{b_\Gamma, p\}} \binom{b_\Gamma}{i} (-1)^i \varphi_{1, \dim \Psi(\Gamma) - p + i}(\Psi(\Gamma)), \end{aligned} \quad (2.11)$$

where for $\Gamma \prec \Delta'$ we set $b_\Gamma = \dim \Gamma - \dim \Psi(\Gamma)$.

Definition 2.9 Let Δ be an n -dimensional integral polytope in $(\mathbb{R}^n, \mathbb{Z}^n)$.

- (i) (see [3, Section 2.3]) We say that Δ is prime if for any vertex w of Δ the cone $\text{Con}(\Delta, w)$ is generated by a basis of \mathbb{R}^n .
- (ii) (see [14, Definition 2.10]) We say that Δ is pseudo-prime if for any 1-dimensional face $\gamma \prec \Delta$ the number of the 2-dimensional faces $\gamma' \prec \Delta$ such that $\gamma \prec \gamma'$ is $n - 1$.

By definition, prime polytopes are pseudo-prime. Moreover any face of a pseudo-prime polytope is again pseudo-prime.

For $\alpha \in \mathbb{C} \setminus \{1\}$ and a face $\Gamma \prec \Delta$, set $\tilde{\varphi}_\alpha(\Gamma) = \sum_{i=0}^{\dim \Gamma} \varphi_{\alpha, i}(\Gamma)$. Then as in [3, Section 5.5 and Theorem 5.6] we obtain the following result.

Proposition 2.10 ([14, Corollary 2.15]) *Assume that $\Delta = NP(g)$ is pseudo-prime. Then for any $\alpha \in \mathbb{C} \setminus \{1\}$ and $r \geq 0$, we have*

$$\sum_{p+q=r} e^{p,q}(Z^*)_{\alpha} = (-1)^{n+r} \sum_{\substack{\Gamma \prec \Delta \\ \dim \Gamma = r+1}} \left\{ \sum_{\Gamma' \prec \Gamma} (-1)^{\dim \Gamma'} \tilde{\varphi}_{\alpha}(\Gamma') \right\}. \quad (2.12)$$

The following lemma will be used later.

Lemma 2.11 *Let γ be a d -dimensional prime polytope. Then for any $0 \leq p \leq d$ we have*

$$\sum_{\Gamma \prec \gamma} (-1)^{\dim \Gamma} \binom{\dim \Gamma}{p} = \sum_{\Gamma \prec \gamma} (-1)^{d+\dim \Gamma} \binom{\dim \Gamma}{d-p}. \quad (2.13)$$

Proof. For a polytope Δ , denote the number of the j -dimensional faces of Δ by $f_{\Delta,j}$ and set $f_{\Delta,-1} = 1$. Let γ^{\vee} be the dual polytope of γ . Then γ^{\vee} is simplicial and we have $f_{\gamma^{\vee},j} = f_{\gamma,d-1-j}$ for any $0 \leq j \leq d$. Hence (2.13) follows from the Dehn-Sommerville equations (see [23] etc.) for simplicial polytopes. \square

3 Motivic Milnor fibers

In [4] and [5] Denef and Loeser introduced motivic Milnor fibers. In this section, we recall their definition and basic properties. Let $f \in \mathbb{C}[x_1, x_2, \dots, x_n]$ be a polynomial such that the hypersurface $f^{-1}(0) = \{x \in \mathbb{C}^n \mid f(x) = 0\}$ has an isolated singular point at $0 \in \mathbb{C}^n$. Then by a fundamental theorem of Milnor [15], for the Milnor fiber F_0 of f at 0 we have $H^j(F_0; \mathbb{C}) \simeq 0$ ($j \neq 0, n-1$). Denote by $\Phi_{n-1,0}: H^{n-1}(F_0; \mathbb{C}) \xrightarrow{\sim} H^{n-1}(F_0; \mathbb{C})$ the $(n-1)$ -th Milnor monodromy of f at $0 \in \mathbb{C}^n$. Let $\pi: X \rightarrow \mathbb{C}^n$ be an embedded resolution of $f^{-1}(0)$ such that $\pi^{-1}(0)$ and $\pi^{-1}(f^{-1}(0))$ are normal crossing divisors in X . Let D_1, D_2, \dots, D_m be the irreducible components of $\pi^{-1}(0)$ and denote by Z the proper transform of $f^{-1}(0)$ in X . For $1 \leq i \leq m$ denote by $a_i > 0$ the order of the zero of $g := f \circ \pi$ along D_i . For a non-empty subset $I \subset \{1, 2, \dots, m\}$ we set $d_I = \gcd(a_i)_{i \in I} > 0$, $D_I = \bigcap_{i \in I} D_i$ and

$$D_I^{\circ} = D_I \setminus \left\{ \left(\bigcup_{i \notin I} D_i \right) \cup Z \right\} \subset X. \quad (3.1)$$

Moreover we set

$$Z_I^{\circ} = \left\{ D_I \setminus \left(\bigcup_{i \notin I} D_i \right) \right\} \cap Z \subset X. \quad (3.2)$$

Then, as in [5, Section 3.3], we can construct an unramified Galois covering $\widetilde{D}_I^{\circ} \rightarrow D_I^{\circ}$ of D_I° as follows. First, for a point $p \in D_I^{\circ}$ we take an affine open neighborhood $W \subset X \setminus \{(\bigcup_{i \notin I} D_i) \cup Z\}$ of p on which there exist regular functions ξ_i ($i \in I$) such that $D_i \cap W = \{\xi_i = 0\}$ for any $i \in I$. Then on W we have $g = f \circ \pi = g_{1,W}(g_{2,W})^{d_I}$, where we set $g_{1,W} = g \prod_{i \in I} \xi_i^{-a_i}$ and $g_{2,W} = \prod_{i \in I} \xi_i^{\frac{a_i}{d_I}}$. Note that $g_{1,W}$ is a unit on W and $g_{2,W}: W \rightarrow \mathbb{C}$ is a regular function. It is easy to see that D_I° is covered by such affine open subsets W . Then as in [5, Section 3.3] by gluing the varieties

$$\widetilde{D_{I,W}^{\circ}} = \{(t, z) \in \mathbb{C}^* \times (D_I^{\circ} \cap W) \mid t^{d_I} = (g_{1,W})^{-1}(z)\} \quad (3.3)$$

together in the following way, we obtain the variety \widetilde{D}_I° over D_I° . If W' is another such open subset and $g = g_{1,W'}(g_{2,W'})^{d_I}$ is the decomposition of g on it, we patch $\widetilde{D}_{I,W}^\circ$ and $\widetilde{D}_{I,W'}^\circ$ by the morphism $(t, z) \mapsto (g_{2,W'}(z)(g_{2,W})^{-1}(z) \cdot t, z)$ defined over $W \cap W'$. Now for $d \in \mathbb{Z}_{>0}$, let $\mu_d \simeq \mathbb{Z}/\mathbb{Z}d$ be the multiplicative group consisting of the d -roots in \mathbb{C} . We denote by $\hat{\mu}$ the projective limit $\varprojlim_d \mu_d$ of the projective system $\{\mu_i\}_{i \geq 1}$ with morphisms

$\mu_{id} \longrightarrow \mu_i$ given by $t \mapsto t^d$. Then the unramified Galois covering \widetilde{D}_I° of D_I° admits a natural μ_{d_I} -action defined by assigning the automorphism $(t, z) \mapsto (\zeta_{d_I} t, z)$ of \widetilde{D}_I° to the generator $\zeta_{d_I} := \exp(2\pi\sqrt{-1}/d_I) \in \mu_{d_I}$. Namely the variety \widetilde{D}_I° is equipped with a good $\hat{\mu}$ -action in the sense of Denef-Loeser [5, Section 2.4]. Note that also the variety Z_I° is equipped with the trivial good $\hat{\mu}$ -action. Following the notations in [5], denote by $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ the ring obtained from the Grothendieck ring $K_0^{\hat{\mu}}(\text{Var}_{\mathbb{C}})$ of varieties over \mathbb{C} with good $\hat{\mu}$ -actions by inverting the Lefschetz motive $\mathbb{L} \simeq \mathbb{C} \in K_0^{\hat{\mu}}(\text{Var}_{\mathbb{C}})$. Recall that $\mathbb{L} \in K_0^{\hat{\mu}}(\text{Var}_{\mathbb{C}})$ is endowed with the trivial action of $\hat{\mu}$.

Definition 3.1 (Denef and Loeser [4] and [5]) We define the motivic Milnor fiber $\mathcal{S}_{f,0} \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ of f at $0 \in \mathbb{C}^n$ by

$$\mathcal{S}_{f,0} = \sum_{I \neq \emptyset} \left\{ (1 - \mathbb{L})^{\#I-1} [\widetilde{D}_I^\circ] + (1 - \mathbb{L})^{\#I} [Z_I^\circ] \right\} \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}. \quad (3.4)$$

As in [5, Section 3.1.2 and 3.1.3], we denote by HS^{mon} the abelian category of Hodge structures with a quasi-unipotent endomorphism. Let $K_0(\text{HS}^{\text{mon}})$ be its Grothendieck ring. Then as in [5], to the cohomology groups $H^j(F_0; \mathbb{C})$ and the semisimple parts of their monodromy automorphisms, we can naturally associate an element

$$[H_f] \in K_0(\text{HS}^{\text{mon}}). \quad (3.5)$$

To describe the element $[H_f] \in K_0(\text{HS}^{\text{mon}})$ in terms of $\mathcal{S}_{f,0} \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$, let

$$\chi_h: \mathcal{M}_{\mathbb{C}}^{\hat{\mu}} \longrightarrow K_0(\text{HS}^{\text{mon}}) \quad (3.6)$$

be the Hodge characteristic morphism defined in [5] which associates to a variety Z with a good μ_d -action the Hodge structure

$$\chi_h([Z]) = \sum_{j \in \mathbb{Z}} (-1)^j [H_c^j(Z; \mathbb{Q})] \in K_0(\text{HS}^{\text{mon}}) \quad (3.7)$$

with the actions induced by the one $z \mapsto \exp(2\pi\sqrt{-1}/d)z$ ($z \in Z$) on Z . Then we have the following fundamental result.

Theorem 3.2 (Denef-Loeser [4, Theorem 4.2.1]) *In the Grothendieck group $K_0(\text{HS}^{\text{mon}})$, we have*

$$[H_f] = \chi_h(\mathcal{S}_{f,0}). \quad (3.8)$$

For $[H_f] \in K_0(\text{HS}^{\text{mon}})$ also the following result due to Steenbrink [24] and Saito [19], [21] is fundamental.

Theorem 3.3 (Steenbrink [24] and Saito [19], [21]) *In the situation as above, we have*

- (i) *Let $\lambda \in \mathbb{C}^* \setminus \{1\}$. Then we have $e^{p,q}([H_f])_\lambda = 0$ for $(p, q) \notin [0, n-1] \times [0, n-1]$. Moreover for $(p, q) \in [0, n-1] \times [0, n-1]$ we have*

$$e^{p,q}([H_f])_\lambda = e^{n-1-q, n-1-p}([H_f])_\lambda. \quad (3.9)$$

- (ii) *We have $e^{p,q}([H_f])_1 = 0$ for $(p, q) \notin \{(0, 0)\} \sqcup ([1, n-1] \times [1, n-1])$ and $e^{0,0}([H_f])_1 = 1$. Moreover for $(p, q) \in [1, n-1] \times [1, n-1]$ we have*

$$e^{p,q}([H_f])_1 = e^{n-q, n-p}([H_f])_1. \quad (3.10)$$

We can check these symmetries of $e^{p,q}([H_f])_\lambda$ by calculating $\chi_h(\mathcal{S}_{f,0}) \in K_0(\text{HS}^{\text{mon}})$ explicitly by our methods (see Section 4) in many cases. Since the weights of $[H_f] \in K_0(\text{HS}^{\text{mon}})$ are defined by the monodromy filtration, we have the following result.

Theorem 3.4 *In the situation as above, we have*

- (i) *Let $\lambda \in \mathbb{C}^* \setminus \{1\}$ and $k \geq 1$. Then the number of the Jordan blocks for the eigenvalue λ with sizes $\geq k$ in $\Phi_{n-1,0}: H^{n-1}(F_0; \mathbb{C}) \xrightarrow{\sim} H^{n-1}(F_0; \mathbb{C})$ is equal to*

$$(-1)^{n-1} \sum_{p+q=n-2+k, n-1+k} e^{p,q}(\chi_h(\mathcal{S}_{f,0}))_\lambda. \quad (3.11)$$

- (ii) *For $k \geq 1$, the number of the Jordan blocks for the eigenvalue 1 with sizes $\geq k$ in $\Phi_{n-1,0}$ is equal to*

$$(-1)^{n-1} \sum_{p+q=n-1+k, n+k} e^{p,q}(\chi_h(\mathcal{S}_{f,0}))_1. \quad (3.12)$$

4 Jordan normal forms of Milnor monodromies

Our methods in [14] can be applied also to the Jordan normal forms of local Milnor monodromies. Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a polynomial such that the hypersurface $\{x \in \mathbb{C}^n \mid f(x) = 0\}$ has an isolated singular point at $0 \in \mathbb{C}^n$.

Definition 4.1 Let $f(x) = \sum_{v \in \mathbb{Z}_+^n} a_v x^v \in \mathbb{C}[x_1, \dots, x_n]$ be a polynomial on \mathbb{C}^n .

- (i) We call the convex hull of $\bigcup_{v \in \text{supp}(f)} \{v + \mathbb{R}_+^n\}$ in \mathbb{R}_+^n the Newton polyhedron of f and denote it by $\Gamma_+(f)$.
- (ii) The union of the compact faces of $\Gamma_+(f)$ is called the Newton boundary of f and denoted by Γ_f .
- (iii) We say that f is convenient if $\Gamma_+(f)$ intersects the positive part of any coordinate axis in \mathbb{R}^n .

Definition 4.2 ([9]) We say that a polynomial $f(x) = \sum_{v \in \mathbb{Z}_+^n} a_v x^v$ ($a_v \in \mathbb{C}$) is non-degenerate at $0 \in \mathbb{C}^n$ if for any face $\gamma \prec \Gamma_+(f)$ such that $\gamma \subset \Gamma_f$ the complex hypersurface $\{x \in (\mathbb{C}^*)^n \mid f_\gamma(x) = 0\}$ in $(\mathbb{C}^*)^n$ is smooth and reduced, where we set $f_\gamma(x) = \sum_{v \in \gamma \cap \mathbb{Z}_+^n} a_v x^v$.

Recall that generic polynomials having a fixed Newton polyhedron are non-degenerate at $0 \in \mathbb{C}^n$. From now on, we always assume also that $f = \sum_{v \in \mathbb{Z}_+^n} a_v x^v \in \mathbb{C}[x_1, \dots, x_n]$ is convenient and non-degenerate at $0 \in \mathbb{C}^n$. For each face $\gamma \prec \Gamma_+(f)$ such that $\gamma \subset \Gamma_f$, let $d_\gamma > 0$ be the lattice distance of γ from the origin $0 \in \mathbb{R}^n$ and Δ_γ the convex hull of $\{0\} \sqcup \gamma$ in \mathbb{R}^n . Let $\mathbb{L}(\Delta_\gamma)$ be the $(\dim \gamma + 1)$ -dimensional linear subspace of \mathbb{R}^n spanned by Δ_γ and consider the lattice $M_\gamma = \mathbb{Z}^n \cap \mathbb{L}(\Delta_\gamma) \simeq \mathbb{Z}^{\dim \gamma + 1}$ in it. Then we set $T_{\Delta_\gamma} := \text{Spec}(\mathbb{C}[M_\gamma]) \simeq (\mathbb{C}^*)^{\dim \gamma + 1}$. Moreover let $\mathbb{L}(\gamma)$ be the smallest affine linear subspace of \mathbb{R}^n containing γ and for $v \in M_\gamma$ define their lattice heights $\text{ht}(v, \gamma) \in \mathbb{Z}$ from $\mathbb{L}(\gamma)$ in $\mathbb{L}(\Delta_\gamma)$ so that we have $\text{ht}(0, \gamma) = d_\gamma > 0$. Then to the group homomorphism $M_\gamma \longrightarrow \mathbb{C}^*$ defined by $v \longmapsto \zeta_{d_\gamma}^{-\text{ht}(v, \gamma)}$ we can naturally associate an element $\tau_\gamma \in T_{\Delta_\gamma}$. We define a Laurent polynomial $g_\gamma = \sum_{v \in M_\gamma} b_v x^v$ on T_{Δ_γ} by

$$b_v = \begin{cases} a_v & (v \in \gamma), \\ -1 & (v = 0), \\ 0 & (\text{otherwise}). \end{cases} \quad (4.1)$$

Then we have $NP(g_\gamma) = \Delta_\gamma$, $\text{supp}(g_\gamma) \subset \{0\} \sqcup \gamma$ and the hypersurface $Z_{\Delta_\gamma}^* = \{x \in T_{\Delta_\gamma} \mid g_\gamma(x) = 0\}$ is non-degenerate by [14, Proposition 5.3]. Moreover $Z_{\Delta_\gamma}^* \subset T_{\Delta_\gamma}$ is invariant by the multiplication $l_{\tau_\gamma}: T_{\Delta_\gamma} \xrightarrow{\sim} T_{\Delta_\gamma}$ by τ_γ , and hence we obtain an element $[Z_{\Delta_\gamma}^*]$ of $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$. Let $\mathbb{L}(\gamma)' \simeq \mathbb{R}^{\dim \gamma}$ be a linear subspace of \mathbb{R}^n such that $\mathbb{L}(\gamma) = \mathbb{L}(\gamma)' + w$ for some $w \in \mathbb{Z}^n$ and set $\gamma' = \gamma - w \subset \mathbb{L}(\gamma)'$. We define a Laurent polynomial $g'_\gamma = \sum_{v \in \mathbb{L}(\gamma)' \cap \mathbb{Z}^n} b'_v x^v$ on $T(\gamma) := \text{Spec}(\mathbb{C}[\mathbb{L}(\gamma)' \cap \mathbb{Z}^n]) \simeq (\mathbb{C}^*)^{\dim \gamma}$ by

$$b'_v = \begin{cases} a_{v+w} & (v \in \gamma'), \\ 0 & (\text{otherwise}). \end{cases} \quad (4.2)$$

Then we have $NP(g'_\gamma) = \gamma'$ and the hypersurface $Z_\gamma^* = \{x \in T(\gamma) \mid g'_\gamma(x) = 0\}$ is non-degenerate. We define $[Z_\gamma^*] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ to be the class of the variety Z_γ^* with the trivial action of $\hat{\mu}$. Finally let $S_\gamma \subset \{1, 2, \dots, n\}$ be the minimal subset S of $\{1, 2, \dots, n\}$ such that $\gamma \subset \{(y_1, y_2, \dots, y_n) \in \mathbb{R}^n \mid y_i = 0 \text{ for any } i \notin S\} \simeq \mathbb{R}^{\#S}$ and set $m_\gamma := \#S_\gamma - \dim \gamma - 1 \geq 0$. Then as in the same way as [14, Theorem 5.7] we obtain the following theorem.

Theorem 4.3 *In the situation as above, we have*

(i) *In the Grothendieck group $K_0(\text{HS}^{\text{mon}})$, we have*

$$\chi_h(\mathcal{S}_{f,0}) = \sum_{\gamma \subset \Gamma_f} \chi_h((1 - \mathbb{L})^{m_\gamma} \cdot [Z_{\Delta_\gamma}^*]) + \sum_{\substack{\gamma \subset \Gamma_f \\ \dim \gamma \geq 1}} \chi_h((1 - \mathbb{L})^{m_\gamma + 1} \cdot [Z_\gamma^*]). \quad (4.3)$$

(ii) *Let $\lambda \in \mathbb{C}^* \setminus \{1\}$ and $k \geq 1$. Then the number of the Jordan blocks for the eigenvalue λ with sizes $\geq k$ in $\Phi_{n-1,0}: H^{n-1}(F_0; \mathbb{C}) \xrightarrow{\sim} H^{n-1}(F_0; \mathbb{C})$ is equal to*

$$(-1)^{n-1} \sum_{p+q=n-2+k, n-1+k} \left\{ \sum_{\gamma \subset \Gamma_f} e^{p,q} \left(\chi_h \left((1 - \mathbb{L})^{m_\gamma} \cdot [Z_{\Delta_\gamma}^*] \right) \right)_\lambda \right\}. \quad (4.4)$$

- (iii) For $k \geq 1$, the number of the Jordan blocks for the eigenvalue 1 with sizes $\geq k$ in $\Phi_{n-1,0}$ is equal to

$$(-1)^{n-1} \sum_{p+q=n-1+k, n+k} \left\{ \sum_{\gamma \in \Gamma_f} e^{p,q} (\chi_h((1-\mathbb{L})^{m_\gamma} \cdot [Z_{\Delta_\gamma}^*]))_1 + \sum_{\substack{\gamma \in \Gamma_f \\ \dim \gamma \geq 1}} e^{p,q} (\chi_h((1-\mathbb{L})^{m_\gamma+1} \cdot [Z_\gamma^*]))_1 \right\}. \quad (4.5)$$

Proof. Since (ii) and (iii) follow from (i) and Theorem 3.4, it suffices to prove (i). The proof is very similar to the one in Varchenko [26]. Let Σ_1 be the dual fan of $\Gamma_+(f)$ in \mathbb{R}_+^n and Σ its smooth subdivision. Denote by X_Σ the smooth toric variety associated to Σ (see Fulton [8] and Oda [16] etc.). Since the union of the cones in Σ is \mathbb{R}_+^n , there exists a proper morphism $\pi: X_\Sigma \rightarrow \mathbb{C}^n$. By the convenience of f , we can construct the smooth fan Σ without subdividing the cones contained in $\partial \mathbb{R}_+^n$ (see [17, Lemma (2.6), Chapter II]). Then π induces an isomorphism $X_\Sigma \setminus \pi^{-1}(0) \simeq \mathbb{C}^n \setminus \{0\}$. Moreover by the non-degeneracy at $0 \in \mathbb{C}^n$ of f , the proper transform Z of the hypersurface $\{x \in \mathbb{C}^n \mid f(x) = 0\}$ in X_Σ is smooth and intersects T -orbits in $\pi^{-1}(0)$ transversally. Let D_1, \dots, D_m be the toric divisors in $\pi^{-1}(0) \subset X_\Sigma$. For a non-empty subset $I \subset \{1, 2, \dots, m\}$ we set $D_I = \bigcap_{i \in I} D_i$ and

$$D_I^\circ = D_I \setminus \left\{ \left(\bigcup_{i \notin I} D_i \right) \cup Z \right\} \subset X_\Sigma \quad (4.6)$$

and define its unramified Galois covering \widetilde{D}_I° as in Section 3. Moreover we set

$$Z_I^\circ = \left\{ D_I \setminus \left(\bigcup_{i \notin I} D_i \right) \right\} \cap Z \subset X_\Sigma \quad (4.7)$$

and denote by $[Z_I^\circ] \in \mathcal{M}_{\mathbb{C}}^\mu$ the class of the variety Z_I° with the trivial action. Then, unlike the global object \mathcal{S}_f^∞ in [14], Denef-Loeser's "local" motivic Milnor fiber $\mathcal{S}_{f,0}$ contains not only $(1-\mathbb{L})^{\sharp I-1}[\widetilde{D}_I^\circ]$ but also $(1-\mathbb{L})^{\sharp I}[Z_I^\circ]$ (see Definition 3.1). These new elements yield the second term in the right hand side of (4.3). Finally, in the Grothendieck group $K_0(\text{HS}^{\text{mon}})$ we can rewrite $\chi_h(\mathcal{S}_{f,0})$ in terms of the dual fan Σ_1 (i.e. in terms of $\Gamma_+(f)$) as in the same way as the proof of [14, Theorem 5.7 (i)]. This completes the proof. \square

Let q_1, \dots, q_l (resp. $\gamma_1, \dots, \gamma_{l'}$) be the 0-dimensional (resp. 1-dimensional) faces of $\Gamma_+(f)$ such that $q_i \in \text{Int}(\mathbb{R}_+^n)$ (resp. $\text{rel.int}(\gamma_i) \subset \text{Int}(\mathbb{R}_+^n)$). Here $\text{rel.int}(\cdot)$ stands for the relative interior. For each q_i (resp. γ_i), denote by $d_i > 0$ (resp. $e_i > 0$) the lattice distance $\text{dist}(q_i, 0)$ (resp. $\text{dist}(\gamma_i, 0)$) of it from the origin $0 \in \mathbb{R}^n$. For $1 \leq i \leq l'$, let Δ_i be the convex hull of $\{0\} \sqcup \gamma_i$ in \mathbb{R}^n . Then for $\lambda \in \mathbb{C} \setminus \{1\}$ and $1 \leq i \leq l'$ such that $\lambda^{e_i} = 1$ we set

$$n(\lambda)_i = \sharp\{v \in \mathbb{Z}^n \cap \text{rel.int}(\Delta_i) \mid \text{ht}(v, \gamma_i) = k\} + \sharp\{v \in \mathbb{Z}^n \cap \text{rel.int}(\Delta_i) \mid \text{ht}(v, \gamma_i) = e_i - k\}, \quad (4.8)$$

where k is the minimal positive integer satisfying $\lambda = \zeta_{e_i}^k$ and for $v \in \mathbb{Z}^n \cap \text{rel.int}(\Delta_i)$ we denote by $\text{ht}(v, \gamma_i)$ the lattice height of v from the base γ_i of Δ_i . As in the same way as [14, Theorem 5.9], by using Propositions 2.5 and 2.6 and Theorem 4.3 (ii), we obtain the following theorem.

Theorem 4.4 *In the situation as above, for $\lambda \in \mathbb{C}^* \setminus \{1\}$, we have*

- (i) *The number of the Jordan blocks for the eigenvalue λ with the maximal possible size n in $\Phi_{n-1,0}$ is equal to $\#\{q_i \mid \lambda^{q_i} = 1\}$.*
- (ii) *The number of the Jordan blocks for the eigenvalue λ with size $n-1$ in $\Phi_{n-1,0}$ is equal to $\sum_{i: \lambda^{e_i}=1} n(\lambda)_i$.*

Note that by Theorem 4.3 and our results in Section 2 we can always calculate the whole Jordan normal form of $\Phi_{n-1,0}$. From now on, we shall rewrite Theorem 4.3 (ii) more explicitly in the case where any face $\gamma \prec \Gamma_+(f)$ such that $\gamma \subset \Gamma_f$ is prime (see Definition 2.9 (i)). Recall that by Proposition 2.3 for $\lambda \in \mathbb{C}^* \setminus \{1\}$ and a face $\gamma \prec \Gamma_+(f)$ such that $\gamma \subset \Gamma_f$ we have $e^{p,q}(Z_{\Delta_\gamma}^*)_\lambda = 0$ for any $p, q \geq 0$ such that $p+q > \dim \Delta_\gamma - 1 = \dim \gamma$. So the non-negative integers $r \geq 0$ such that $\sum_{p+q=r} e^{p,q}(Z_{\Delta_\gamma}^*)_\lambda \neq 0$ are contained in the closed interval $[0, \dim \gamma] \subset \mathbb{R}$.

Definition 4.5 For a face $\gamma \prec \Gamma_+(f)$ such that $\gamma \subset \Gamma_f$ and $k \geq 1$, we define a finite subset $J_{\gamma,k} \subset [0, \dim \gamma] \cap \mathbb{Z}$ by

$$J_{\gamma,k} = \{0 \leq r \leq \dim \gamma \mid n-2+k \equiv r \pmod{2}\}. \quad (4.9)$$

For each $r \in J_{\gamma,k}$, set

$$d_{k,r} = \frac{n-2+k-r}{2} \in \mathbb{Z}_+. \quad (4.10)$$

If a face $\gamma \prec \Gamma_+(f)$ such that $\gamma \subset \Gamma_f$ is prime, then the polytope Δ_γ is pseudo-prime (see Definition 2.9 (ii)). Then by Proposition 2.10 for $\lambda \in \mathbb{C}^* \setminus \{1\}$ and an integer $r \geq 0$ such that $r \in [0, \dim \gamma]$ we have

$$\sum_{p+q=r} e^{p,q}(\chi_h([Z_{\Delta_\gamma}^*]))_\lambda = (-1)^{\dim \gamma + r + 1} \sum_{\substack{\Gamma \prec \Delta_\gamma \\ \dim \Gamma = r+1}} \left\{ \sum_{\Gamma' \prec \Gamma} (-1)^{\dim \Gamma'} \tilde{\varphi}_\lambda(\Gamma') \right\}. \quad (4.11)$$

For simplicity, we denote this last integer by $e(\gamma, \lambda)_r$. Then by Theorem 4.3 (ii) we obtain the following result.

Theorem 4.6 *Assume that any face $\gamma \prec \Gamma_+(f)$ such that $\gamma \subset \Gamma_f$ is prime. Let $\lambda \in \mathbb{C}^* \setminus \{1\}$ and $k \geq 1$. Then the number of the Jordan blocks for the eigenvalue λ with sizes $\geq k$ in $\Phi_{n-1,0}: H^{n-1}(F_0; \mathbb{C}) \xrightarrow{\sim} H^{n-1}(F_0; \mathbb{C})$ is equal to*

$$(-1)^{n-1} \sum_{\gamma \subset \Gamma_f} \left\{ \sum_{r \in J_{\gamma,k}} (-1)^{d_{k,r}} \binom{m_\gamma}{d_{k,r}} \cdot e(\gamma, \lambda)_r + \sum_{r \in J_{\gamma,k+1}} (-1)^{d_{k+1,r}} \binom{m_\gamma}{d_{k+1,r}} \cdot e(\gamma, \lambda)_r \right\}, \quad (4.12)$$

where we used the convention $\binom{a}{b} = 0$ ($0 \leq a < b$) for binomial coefficients.

By combining the proof of [3, Theorem 5.6] and [14, Proposition 2.14] with Theorem 4.3 (iii), if any face $\gamma \prec \Gamma_+(f)$ such that $\gamma \subset \Gamma_f$ is prime we can also describe the Jordan blocks for the eigenvalue 1 in $\Phi_{n-1,0}$ by a closed formula. Since this result is rather involved, we omit it here.

Remark 4.7 Our results above are different from the previous ones due to Danilov [2] and Tanabé [25]. For example, in [2] and [25] they assume a stronger condition that the Newton polyhedron $\Gamma_+(f)$ itself is prime. We could weaken their condition, because our [14, Propositions 2.13 and 2.14] and Proposition 2.10 are generalizations of the corresponding results in [3] to pseudo-prime polytopes.

We can also obtain the corresponding results for the eigenvalue 1 by rewriting Theorem 4.3 (iii) more simply as follows.

Theorem 4.8 *In the situation of Theorem 4.3, for $k \geq 1$ the number of the Jordan blocks for the eigenvalue 1 with sizes $\geq k$ in $\Phi_{n-1,0}$ is equal to*

$$(-1)^{n-1} \sum_{p+q=n-2-k, n-1-k} \left\{ \sum_{\gamma \subset \Gamma_f} e^{p,q} \left(\chi_h \left((1 - \mathbb{L})^{m_\gamma} \cdot [Z_{\Delta_\gamma}^*] \right) \right)_1 \right\}. \quad (4.13)$$

As in the same way as [14, Theorems 5.11 and 5.12], by using Propositions 2.5 and 2.6 and Theorem 4.8, we obtain the following corollary. Denote by Π_f the number of the lattice points on the 1-skeleton of $\Gamma_f \cap \text{Int}(\mathbb{R}_+^n)$. Also, for a compact face $\gamma \prec \Gamma_+(f)$ we denote by $l^*(\gamma)$ the number of the lattice points on $\text{rel.int}(\gamma)$.

Corollary 4.9 *In the situation as above, we have*

- (i) (van Doorn-Steenbrink [6]) *The number of the Jordan blocks for the eigenvalue 1 with the maximal possible size $n - 1$ in $\Phi_{n-1,0}$ is Π_f .*
- (ii) *The number of the Jordan blocks for the eigenvalue 1 with size $n - 2$ in $\Phi_{n-1,0}$ is equal to $2 \sum_{\gamma} l^*(\gamma)$, where γ ranges through the compact faces of $\Gamma_+(f)$ such that $\dim \gamma = 2$ and $\text{rel.int}(\gamma) \subset \text{Int}(\mathbb{R}_+^n)$.*

Note that Corollary 4.9 (i) was previously obtained in van Doorn-Steenbrink [6] by different methods. Theorem 4.8 asserts that by replacing $\Gamma_+(f)$ with the Newton polyhedron at infinity $\Gamma_\infty(f)$ in [11], [13] and [14] etc. the combinatorial description of the local monodromy $\Phi_{n-1,0}$ is the same as that of the global one Φ_{n-1}^∞ obtained in [14, Theorem 5.7 (iii)]. Namely we find a beautiful symmetry between local and global. Theorem 4.8 can be deduced from the following more precise result.

Theorem 4.10 *In the situation as above, for any $0 \leq p, q \leq n - 2$ we have*

$$\begin{aligned} & \sum_{\gamma \subset \Gamma_f} e^{p,q} \left(\chi_h \left((1 - \mathbb{L})^{m_\gamma} [Z_{\Delta_\gamma}^*] \right) \right)_1 \\ &= \sum_{\gamma \subset \Gamma_f} e^{p+1,q+1} \left(\chi_h \left((1 - \mathbb{L})^{m_\gamma} [Z_{\Delta_\gamma}^*] + (1 - \mathbb{L})^{m_\gamma+1} [Z_\gamma^*] \right) \right)_1. \end{aligned} \quad (4.14)$$

We can easily see that Theorem 4.10 follows from Proposition 4.11 below. For $[V] \in K_0(\text{HS}^{\text{mon}})$, let $e([V])_1 = \sum_{p,q=0}^\infty e^{p,q}([V])_1 t_1^p t_2^q$ be the generating function of $e^{p,q}([V])_1$ as in [3].

Proposition 4.11 *We have*

$$\sum_{\gamma \in \Gamma_f} e \left(\chi_h \left((1 - \mathbb{L})^{m_\gamma+1} ([Z_{\Delta_\gamma}^*] + [Z_\gamma^*]) \right) \right)_1 = 1 - (t_1 t_2)^n. \quad (4.15)$$

From now on, we shall prove Proposition 4.11. First, we apply Proposition 2.8 to the case where $\Delta = \Delta_\gamma$ for a face γ of $\Gamma_+(f)$ such that $\gamma \subset \Gamma_f$. Let γ' be a prime polytope in $\mathbb{R}^{\dim \gamma}$ which majorizes γ and consider the Minkowski sum $\gamma'' := \gamma + \gamma'$ (resp. $\square_{\gamma''} := \Delta_\gamma + \gamma'$) in $\mathbb{R}^{\dim \gamma}$ (resp. $\mathbb{R}^{\dim \gamma+1}$). Then $\square_{\gamma''}$ is a $(\dim \gamma + 1)$ -dimensional truncated pyramid whose top (resp. bottom) is γ' (resp. γ'') (see Figure 1 below). In particular, $\square_{\gamma''}$ is prime. Since the dual fan of γ'' coincides with that of γ' , the prime polytope γ'' majorizes γ . Let $\Psi: \text{som}(\gamma'') \rightarrow \text{som}(\gamma)$ be the morphism between the sets of the vertices of γ'' and γ . By extending Ψ to a morphism $\tilde{\Psi}: \text{som}(\square_{\gamma''}) \rightarrow \text{som}(\Delta_\gamma)$ as

$$\tilde{\Psi}(w) = \begin{cases} \Psi(w) & (w \in \text{som}(\gamma'')), \\ \{0\} & (w \in \text{som}(\gamma')), \end{cases} \quad (4.16)$$

we see that the prime polytope $\square_{\gamma''}$ majorizes Δ_γ .

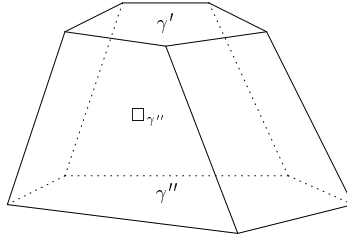


Figure 1

Proposition 4.12 *For the closure $\overline{Z_{\Delta_\gamma}^*}$ of $Z_{\Delta_\gamma}^*$ in $X_{\square_{\gamma''}}$, we have*

$$\sum_q e^{p,q}(\overline{Z_{\Delta_\gamma}^*})_1 = \sum_{\tau \prec \gamma''} (-1)^{\dim \tau + p} \binom{\dim \tau}{p}. \quad (4.17)$$

Proof. It suffices to rewrite Proposition 2.8 in this case. For a face Γ of $\square_{\gamma''}$, we set $b_\Gamma = \dim \Gamma - \dim \tilde{\Psi}(\Gamma)$. Note that the set of faces of $\square_{\gamma''}$ consists of those of γ' and γ'' and side faces. Each side face of $\square_{\gamma''}$ is a truncated pyramid \square_τ whose bottom is $\tau \prec \gamma''$. Since $\dim \square_\tau = \dim \tau + 1$ and $b_{\square_\tau} = b_\tau$ for $\tau \prec \gamma''$, we have

$$\sum_{\Gamma \prec \square_{\gamma''}} (-1)^{\dim \Gamma + p + 1} \left\{ \binom{\dim \Gamma}{p+1} - \binom{b_\Gamma}{p+1} \right\} = \sum_{\tau \prec \gamma''} (-1)^{\dim \tau + p} \binom{\dim \tau}{p} \quad (4.18)$$

and

$$\begin{aligned} & \sum_{\Gamma \prec \square_{\gamma''}} (-1)^{\dim \Gamma + 1} \sum_{i=0}^{\min\{b_\Gamma, p\}} \binom{b_\Gamma}{i} (-1)^i \varphi_{1, \dim \tilde{\Psi}(\Gamma) - p + i}(\tilde{\Psi}(\Gamma)) \\ &= \sum_{\tau \prec \gamma''} (-1)^{\dim \tau + 1} \sum_{i=0}^{\min\{b_\tau, p\}} \binom{b_\tau}{i} (-1)^i \\ & \quad \times \left\{ \varphi_{1, \dim \Psi(\tau) - p + i}(\Psi(\tau)) - \varphi_{1, \dim \tilde{\Psi}(\square_\tau) - p + i}(\tilde{\Psi}(\square_\tau)) \right\}, \end{aligned} \quad (4.19)$$

where the faces τ of the top γ' of $\square_{\gamma''}$ are neglected by the condition $\dim \tilde{\Psi}(\tau) = 0$. By $\tilde{\Psi}(\square_\tau) = \Delta_{\Psi(\tau)}$ and Lemma 4.13 below, the last term is equal to 0. \square

Lemma 4.13 *For any face γ of $\Gamma_+(f)$ such that $\gamma \subset \Gamma_f$, we have*

$$\varphi_{1,j+1}(\Delta_\gamma) = \varphi_{1,j}(\gamma). \quad (4.20)$$

Proof. By the relation $l^*((k+1)\Delta_\gamma)_1 - l^*(k\Delta_\gamma)_1 = l^*(k\gamma)_1$ ($k \geq 0$) we have

$$P_1(\Delta_\gamma; t) = tP_1(\gamma; t). \quad (4.21)$$

By comparing the coefficients of t^{j+1} in both sides, we obtain (4.20). \square

The following proposition is a key in the proof of Proposition 4.11.

Proposition 4.14 *For any face γ of $\Gamma_+(f)$ such that $\gamma \subset \Gamma_f$, we have*

$$e(\chi_h([Z_{\Delta_\gamma}^*] + [Z_\gamma^*]))_1 = (t_1 t_2 - 1)^{\dim \gamma}. \quad (4.22)$$

Proof. It is enough to prove

$$e^{p,q}(Z_\gamma^*)_1 + e^{p,q}(Z_{\Delta_\gamma}^*)_1 = (-1)^{\dim \gamma + p} \binom{\dim \gamma}{p} \cdot \delta_{p,q}, \quad (4.23)$$

where $\delta_{p,q}$ is Kronecker's delta. We consider the closure $\overline{Z_{\Delta_\gamma}^*}$ of $Z_{\Delta_\gamma}^*$ in $X_{\square_{\gamma''}}$. Then by the proofs of Propositions 2.8 and 4.12, we have

$$e^{p,q}(\overline{Z_{\Delta_\gamma}^*})_1 = \sum_{\tau \prec \gamma''} \left\{ e^{p,q}((\mathbb{C}^*)^{b_\tau} \times Z_{\Psi(\tau)}^*)_1 + e^{p,q}((\mathbb{C}^*)^{b_{\square_\tau}} \times Z_{\tilde{\Psi}(\square_\tau)}^*)_1 \right\} \quad (4.24)$$

$$= \sum_{\tau \prec \gamma''} \sum_{i=0}^{\min\{b_\tau, p\}} \binom{b_\tau}{i} (-1)^{i+b_\tau} \left\{ e^{p-i, q-i}(Z_{\Psi(\tau)}^*)_1 + e^{p-i, q-i}(Z_{\Delta_{\Psi(\tau)}}^*)_1 \right\}. \quad (4.25)$$

Let us prove (4.23) by induction on $\dim \gamma$. In the case $\dim \gamma = 0$, we can prove (4.23) easily by Propositions 2.3 and 2.6. Assume that for any $\sigma \subset \Gamma_f$ such that $\dim \sigma < \dim \gamma$ (4.23) holds. Then by $b_{\gamma''} = 0$ and (4.25) we have

$$e^{p,q}(\overline{Z_{\Delta_\gamma}^*})_1 = e^{p,q}(Z_\gamma^*)_1 + e^{p,q}(Z_{\Delta_\gamma}^*)_1 + \delta_{p,q} \sum_{\tau \not\prec \gamma''} (-1)^{\dim \tau + p} \binom{\dim \tau}{p}. \quad (4.26)$$

In the case $p + q > \dim \gamma$, by Proposition 2.3 we have

$$e^{p,q}(\overline{Z_{\Delta_\gamma}^*})_1 = \delta_{p,q} \sum_{\tau \prec \gamma''} (-1)^{\dim \tau + p} \binom{\dim \tau}{p}. \quad (4.27)$$

Therefore, also in the case $p + q < \dim \gamma$, by the Poincaré duality for $\overline{Z_{\Delta_\gamma}^*}$ ($\square_{\gamma''}$ is prime) and Lemma 2.11 we have

$$e^{p,q}(\overline{Z_{\Delta_\gamma}^*})_1 = e^{\dim \gamma - p, \dim \gamma - q}(\overline{Z_{\Delta_\gamma}^*})_1 \quad (4.28)$$

$$= \delta_{p,q} \sum_{\tau \prec \gamma''} (-1)^{\dim \tau + \dim \gamma - p} \binom{\dim \tau}{\dim \gamma - p} \quad (4.29)$$

$$= \delta_{p,q} \sum_{\tau \prec \gamma''} (-1)^{\dim \tau + p} \binom{\dim \tau}{p}. \quad (4.30)$$

In the case $p + q = \dim \gamma$, by Proposition 4.12 and the previous results we have

$$e^{p,q}(\overline{Z_{\Delta_\gamma}^*})_1 = \sum_{q'} e^{p,q'}(\overline{Z_{\Delta_\gamma}^*})_1 - (1 - \delta_{p,q})e^{p,p}(\overline{Z_{\Delta_\gamma}^*})_1 \quad (4.31)$$

$$= \delta_{p,q} \sum_{\tau \prec \gamma''} (-1)^{\dim \tau + p} \binom{\dim \tau}{p}. \quad (4.32)$$

By (4.26), we obtain (4.23) for any p, q . \square

Now we can finish the proof of Proposition 4.11 as follows. By Proposition 4.14, we have

$$\sum_{\gamma \in \Gamma_f} e \left(\chi_h \left((1 - \mathbb{L})^{m_\gamma + 1} ([Z_{\Delta_\gamma}^*] + [Z_\gamma^*]) \right) \right)_1 = \sum_{\gamma \in \Gamma_f} (1 - t_1 t_2)^{m_\gamma + 1} (t_1 t_2 - 1)^{\dim \gamma} \quad (4.33)$$

$$= \sum_{l=1}^n (1 - t_1 t_2)^l \sum_{\#S_\gamma=l} (-1)^{\dim \gamma} \quad (4.34)$$

$$= \sum_{l=1}^n (1 - t_1 t_2)^l \binom{n}{l} (-1)^{l-1} \quad (4.35)$$

$$= 1 - (t_1 t_2)^n. \quad (4.36)$$

\square

Remark 4.15 Following the proof of [14, Theorem 5.16], we can easily give another proof to the Steenbrink conjecture which was proved by Varchenko-Khovanskii [27] and Saito [20] independently. For an introduction to this conjecture, see an excellent survey in Kulikov [10] etc.

Remark 4.16 For a polynomial map $f: \mathbb{C}^n \rightarrow \mathbb{C}$, it is well-known that there exists a finite subset $B \subset \mathbb{C}$ such that the restriction

$$\mathbb{C}^n \setminus f^{-1}(B) \rightarrow \mathbb{C} \setminus B \quad (4.37)$$

of f is a locally trivial fibration. We denote by B_f the smallest such subset $B \subset \mathbb{C}$. For a point $b \in B_f$, take a small circle $C_\varepsilon(b) = \{x \in \mathbb{C} \mid |x - b| = \varepsilon\}$ ($0 < \varepsilon \ll 1$) around b such that $B_f \cap \{x \in \mathbb{C} \mid |x - b| \leq \varepsilon\} = \{b\}$. Then by the restriction of $\mathbb{C}^n \setminus f^{-1}(B_f) \rightarrow \mathbb{C} \setminus B_f$ to $C_\varepsilon(b) \subset \mathbb{C} \setminus B_f$ we obtain a geometric monodromy automorphism $\Phi_f^b: f^{-1}(b + \varepsilon) \xrightarrow{\sim} f^{-1}(b + \varepsilon)$ and the linear maps

$$\Phi_f^b: H^j(f^{-1}(b + \varepsilon); \mathbb{C}) \xrightarrow{\sim} H^j(f^{-1}(b + \varepsilon); \mathbb{C}) \quad (j = 0, 1, \dots) \quad (4.38)$$

associated to it. The eigenvalues of Φ_f^b were studied in [13, Sections 3 and 4] etc. If f is tame at infinity, as in [14, Section 4] we can introduce a motivic Milnor fiber $\mathcal{S}_f^b \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ along the central fiber $f^{-1}(b)$ to calculate the numbers of the Jordan blocks for the eigenvalues $\lambda \neq 1$ in Φ_{n-1}^b . This result can be easily obtained by using the proof of Sabbah [18, Theorem 13.1]. It would be an interesting problem to construct a motivic object to calculate the eigenvalue 1 part of Φ_{n-1}^b .

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